

Notes on Optimizing FL Programs

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1. Introduction

The following describes a functional, $\text{Gen}:\langle p, f, g, h \rangle$, that corresponds to a simple loop for computing a value by iteratively modifying two quantities, x and y , by applying the function f to x to get the new x , and g to $\langle h:x, y \rangle$ to get the new y . The final value is y when $p:x$ becomes true.

When $g=a1$, $\text{Gen}:\langle p, f, a1, h \rangle \circ [\text{id}, []] : x$ computes, in succession,

$$\langle x, \langle \rangle \rangle \rightarrow_{\text{Gen}} \langle f:x, \langle h:x \rangle \rangle \rightarrow_{\text{Gen}} \langle f^2:x, \langle h \circ f:x, h:x \rangle \rangle \rightarrow_{\text{Gen}} \dots$$

yielding

$$\langle h \circ f^n x, \dots, h \circ f:x, h:x \rangle \text{ when } p \circ f^{n+1}:x \text{ is true}$$

The main result is that

$$/g \circ \text{Gen}:\langle p, f, a1, h \rangle \circ [\text{id}, []] \equiv \text{Gen}:\langle p, f, g, h \rangle \circ [\text{id}, \sim u]$$

where u is a right unit for g . This means that instead of building a sequence with $\text{Gen}:\langle p, f, a1, h \rangle$ and then reducing it with $/g$, the result can be formed directly with $\text{Gen}:\langle p, f, g, h \rangle$.

1.1 Examples of use of Gen for defining functions:

$$\text{fact} \equiv /* \circ \text{Gen}:\langle \text{eq0}, \text{sub1}, a1, \text{id} \rangle \circ [\text{id}, []]$$

Here $\text{Gen}:\langle \text{eq0}, \text{sub1}, a1, \text{id} \rangle \circ [\text{id}, []]$ is just `ints_to`.

$$\text{len} \equiv /+ \circ \text{Gen}:\langle \text{isnull}, \text{tl}, a1, \sim 1 \rangle \circ [\text{id}, []]$$

2. The Details

Def $\text{Gen} \Leftarrow [p., f., g., h.] \leftarrow [x., y.] \equiv p \circ x \rightarrow y; \text{Gen} : \langle p, f, g, h \rangle \circ [f \circ x, g \circ [h \circ x, y]]$

Lemma 1 $/g \circ [a_1, \dots, a_{n-2}, g \circ [a_{n-1}, a_n]] \equiv /g \circ [a_1, \dots, a_n]$

Proof The LHS expands to

$g \circ [a_1, g \circ [a_2, \dots g \circ [a_{n-2}, g \circ [a_{n-1}, a_n]] \dots]]$

which reduces to the RHS. $\square$

Lemma 2 $[f \circ s_1, g \circ [h \circ s_1, s_2]]^n \equiv$
 $[f^n \circ s_1, /g \circ [h \circ f^{n-1} \circ s_1, \dots, h \circ f \circ s_1, h \circ s_1, s_2]]$

Proof

True for $n=1$. Assume true for $n \geq 1$, prove for $n+1$.

$[f \circ s_1, g \circ [h \circ s_1, s_2]]^{n+1} \equiv [f \circ s_1, g \circ [h \circ s_1, s_2]]^n \circ [f \circ s_1, g \circ [h \circ s_1, s_2]]$

by definition of exponentiation

$\equiv [f^n \circ s_1, /g \circ [h \circ f^{n-1} \circ s_1, \dots, h \circ f \circ s_1, h \circ s_1, s_2]] \circ [f \circ s_1, g \circ [h \circ s_1, s_2]]$

by inductive assumption

$\equiv [f^{n+1} \circ s_1, /g \circ [h \circ f^n \circ s_1, \dots, h \circ f^2 \circ s_1, h \circ f \circ s_1, g \circ [h \circ s_1, s_2]]]$

distributing from right and simplifying; both sides undefined if any part is undefined

$\equiv [f^{n+1} \circ s_1, /g \circ [h \circ f^n \circ s_1, \dots, h \circ f \circ s_1, h \circ s_1, s_2]]$

by lemma 1 $\square$

Lemma 3 $\text{Gen} : \langle p, f, g, h \rangle \equiv p \circ s_1 \rightarrow s_2; \dots ;$
 $p \circ f^n \circ s_1 \rightarrow /g \circ [h \circ f^{n-1} \circ s_1, \dots, h \circ f \circ s_1, h \circ s_1, s_2]; \dots$

Proof By the definition of Gen and the linear expansion theorem:

$\text{Gen} : \langle p, f, g, h \rangle \equiv p \circ s_1 \rightarrow s_2; \dots ;$

$p \circ s_1 \circ [f \circ s_1, g \circ [h \circ s_1, s_2]]^n \rightarrow s_2 \circ [f \circ s_1, g \circ [h \circ s_1, s_2]]^n ; \dots$

substituting for $[f \circ s_1, g \circ [h \circ s_1, s_2]]^n$ from lemma 2 and simplifying gives the desired result; note that if any quantities are undefined, both sides are undefined, thus it is ok to use $s_1 \circ [x, y] \equiv x$, etc. $\square$

Theorem For all functions p, f, g, h , the following holds provided that g has a right unit u such that $g \circ [r, \sim u] \equiv r \leftrightarrow \text{defined} \circ r$ for all functions r :

$$/g \circ \text{Gen}:\langle p, f, a1, h \rangle \circ [id, \overset{\tau}{\underset{\wedge}{[]}}] \equiv \text{Gen}:\langle p, f, g, h \rangle \circ [id, \overset{\dagger}{\sim u}]$$

Proof From lemma 3 we have, after simplifying,

$$\begin{aligned} \text{Gen}:\langle p, f, g, h \rangle \circ [id, w] &\equiv p \rightarrow w; \dots ; \\ p \circ f^n &\rightarrow /g \circ [h \circ f^{n-1}, \dots, h \circ f, h, w]; \dots \end{aligned}$$

when $g=a1$ and $w=[]$, this gives

$$\begin{aligned} \text{Gen}:\langle p, f, a1, h \rangle \circ [id, []] &\equiv p \rightarrow []; \dots ; \\ p \circ f^n &\rightarrow /a1 \circ [h \circ f^{n-1}, \dots, h \circ f, h, []]; \dots \\ &\equiv p \rightarrow []; \dots ; \\ p \circ f^n &\rightarrow [h \circ f^{n-1}, \dots, h \circ f, h]; \dots \end{aligned}$$

Thus

$$\begin{aligned} /g \circ \text{Gen}:\langle p, f, a1, h \rangle \circ [id, []] &\equiv p \rightarrow \sim u; \dots ; \\ p \circ f^n &\rightarrow /g \circ [h \circ f^{n-1}, \dots, h \circ f, h]; \dots \end{aligned}$$

since $/g \circ [] \equiv \sim u$ by definition. And

$$\begin{aligned} \text{Gen}:\langle p, f, g, h \rangle \circ [id, \sim u] &\equiv p \rightarrow \sim u; \dots ; \\ p \circ f^n &\rightarrow /g \circ [h \circ f^{n-1}, \dots, h \circ f, h]; \dots \end{aligned}$$

since $/g \circ [h \circ f^{n-1}, \dots, h \circ f, h, \sim u] \equiv /g \circ [h \circ f^{n-1}, \dots, h \circ f, h]$

□