

Notes on Optimizing FL Programs

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Introduction

The following describes a functional, $\text{Gen}:\langle p, f, g \rangle$, that corresponds to a simple loop for computing a value by iteratively modifying two quantities, x and y , by applying the function f to x to get the new x , and g to $\langle x, y \rangle$ to get the new y . The final value is y when $p:x$ becomes true. The definition of Gen is:

Def $\text{Gen} \Leftarrow [p., f., g.] \leftarrow [x., y.] \equiv p \circ x \rightarrow y; \text{Gen}:\langle p, f, g \rangle \circ [f \circ x, g]$

Thus Gen keeps transforming $\langle x, y \rangle$ into $\langle f:x, g:\langle x, y \rangle \rangle$ until $p:x$ is true. When $g = \text{al} \circ [h \circ s1, s2]$, $\text{Gen}:\langle p, f, g \rangle \circ [\text{id}, []] : x$ computes, in succession,

$$\langle x, \langle \rangle \rangle \rightarrow_{\text{Gen}} \langle f:x, \langle h:x \rangle \rangle \rightarrow_{\text{Gen}} \langle f^2:x, \langle h \circ f:x, h:x \rangle \rangle \rightarrow_{\text{Gen}} \dots$$

yielding

$$\langle h \circ f^{n-1}:x, \dots, h \circ f:x, h:x \rangle \text{ when } p \circ f^n:x \text{ is true}$$

The main result is that, for all p, f, h, w, r, t ,

$$/w \circ \text{Gen}:\langle p, f, \text{al} \circ [h \circ s1, s2] \rangle \circ [r, [t]] \equiv \text{Gen}:\langle p, f, w \circ [h \circ s1, s2] \rangle \circ [r, t]$$

If $h = \text{id}$, this reduces to:

$$/w \circ \text{Gen}:\langle p, f, \text{al} \rangle \circ [r, [t]] \equiv \text{Gen}:\langle p, f, w \rangle \circ [r, t]$$

This means that instead of building a sequence with $\text{Gen}:\langle p, f, \text{al} \circ [h \circ s1, s2] \rangle$ and then reducing it with $/w$, the result can be formed directly with $\text{Gen}:\langle p, f, w \circ [h \circ s1, s2] \rangle$.

Examples of use of Gen for defining functions:

$$\text{fact} \equiv /* \circ \text{Gen}:\langle \text{eq0}, \text{sub1}, \text{al} \rangle \circ [\text{sub1}, [\text{id}]]$$

Here $\text{Gen}:\langle \text{eq0}, \text{sub1}, \text{al} \rangle \circ [\text{sub1}, [\text{id}]]$ is just `ints_to`.

$$\text{len} \equiv /+ \circ \text{Gen}:\langle \text{isnull}, \text{tl}, \text{al}, \sim 1 \rangle \circ [\text{id}, []]$$

here Gen produces the sequence of all 1's of the same length as the argument.

The Details

$$\text{Def } \text{Gen} \leftarrow [p., f., g.] \leftarrow [x., y.] \equiv p \circ x \rightarrow y; \text{Gen}:\langle p, f, g, \rangle \circ [f \circ x, g]$$

$$\text{Lemma 1 } /g \circ [a_1, \dots, a_{n-2}, g \circ [a_{n-1}, a_n]] \equiv /g \circ [a_1, \dots, a_n]$$

Proof The LHS expands to

$$g \circ [a_1, g \circ [a_2, \dots, g \circ [a_{n-2}, g \circ [a_{n-1}, a_n]] \dots]]$$

which reduces to the RHS. □

$$\text{Lemma 2 } [f \circ s1, g]^n \equiv [f^n \circ s1, /g \circ [f^{n-1} \circ s1, \dots, f \circ s1, g]]$$

Proof

True for $n=1$. Assume true for $n \geq 1$, prove for $n+1$.

$$[f \circ s1, g]^{n+1} \equiv [f \circ s1, g]^n \circ [f \circ s1, g]$$

by definition of exponentiation

$$\equiv [f^n \circ s1, /g \circ [f^{n-1} \circ s1, \dots, f \circ s1, g]] \circ [f \circ s1, g]$$

by inductive assumption

$$\equiv [f^{n+1} \circ s1, /g \circ [f^n \circ s1, \dots, f^2 \circ s1, g \circ [f \circ s1, g]]]$$

distributing from right and simplifying; both sides undefined if any part is undifined

$$\equiv [f^{n+1} \circ s1, /g \circ [f^n \circ s1, \dots, f \circ s1, g]]$$

by lemma 1 □

Lemma 3 $\text{Gen}:\langle p, f, g \rangle \equiv p \circ s_1 \rightarrow s_2; \dots ;$
 $p \circ f^n \circ s_1 \rightarrow /g \circ [f^{n-1} \circ s_1, \dots, f \circ s_1, g]; \dots$

Proof By the definition of Gen and the linear expansion theorem:

$$\begin{aligned} \text{Gen}:\langle p, f, g \rangle &\equiv p \circ s_1 \rightarrow s_2; \dots ; \\ p \circ s_1 \circ [f \circ s_1, g]^n &\rightarrow s_2 \circ [f \circ s_1, g]^n ; \dots \end{aligned}$$

substituting for $[f \circ s_1, g]^n$ from lemma 2 and simplifying gives the desired result; note that if any quantities are undefined, both sides are undefined, thus it is ok to use $s_1 \circ [x, y] \equiv x$, etc. $\square$

Lemma 4 $\text{Gen}:\langle p, f, g \circ [h \circ s_1, s_2] \rangle \circ [r, s] \equiv p \circ r \rightarrow s; \dots ;$
 $p \circ f^n \circ r \rightarrow /g \circ [h \circ f^{n-1} \circ r, h \circ f^{n-2} \circ r, \dots, h \circ f \circ r, h \circ r, s]; \dots$

Proof

$$\begin{aligned} \text{Gen}:\langle p, f, g \circ [h \circ s_1, s_2] \rangle \circ [r, s] &\equiv p \circ r \rightarrow s; \dots ; \\ p \circ f^n \circ r &\rightarrow /(g \circ [h \circ s_1, s_2]) \circ [f^{n-1} \circ r, \dots, f \circ r, g \circ [h \circ r, s]]; \dots \\ &\quad \text{by lemma 3} \\ &\equiv p \circ r \rightarrow s; \dots ; \\ p \circ f^n \circ r &\rightarrow g \circ [h \circ f^{n-1} \circ r, g \circ [h \circ f^{n-2} \circ r, \dots, g \circ [h \circ f \circ r, g \circ [h \circ r, s]] \dots]]; \dots \\ &\quad \text{by definition of / and simplification} \\ &\equiv p \circ r \rightarrow s; \dots ; \\ p \circ f^n \circ r &\rightarrow /g \circ [h \circ f^{n-1} \circ r, h \circ f^{n-2} \circ r, \dots, h \circ f \circ r, h \circ r, s]; \dots \\ &\quad \text{by definition of /} \quad \square \end{aligned}$$

Theorem For all functions p, f, h, w, r, t the following holds:

$$/w \circ \text{Gen}:\langle p, f, a \circ [h \circ s_1, s_2] \rangle \circ [r, [t]] \equiv \text{Gen}:\langle p, f, w \circ [h \circ s_1, s_2] \rangle \circ [r, t]$$

Proof From lemma 4 we have, when $s=[t]$,

$$\begin{aligned} \text{Gen}:\langle p, f, g \circ [h \circ s_1, s_2] \rangle \circ [r, [t]] &\equiv p \circ r \rightarrow [t]; \dots ; \\ p \circ f^n \circ r &\rightarrow /g \circ [h \circ f^{n-1} \circ r, h \circ f^{n-2} \circ r, \dots, h \circ f \circ r, h \circ r, [t]]; \dots \end{aligned}$$

when $g=a$ this gives

$$\text{Gen}:\langle p, f, a \circ [h \circ s_1, s_2] \rangle \circ [r, [t]] \equiv p \circ r \rightarrow [t]; \dots ;$$

$$\begin{aligned}
& p \circ f^n \circ r \rightarrow /a1 \circ [h \circ f^{n-1} \circ r, \dots, h \circ f \circ r, h \circ r, [t]]; \dots \\
& \equiv p \circ r \rightarrow [t]; \dots ; \\
& p \circ f^n \circ r \rightarrow [h \circ f^{n-1} \circ r, \dots, h \circ f \circ r, h \circ r, t]; \dots \\
& \qquad \qquad \text{by def of } / \text{ and } a1
\end{aligned}$$

Thus

$$\begin{aligned}
/w \circ \text{Gen}: \langle p, f, a1 \circ [h \circ s1, s2] \rangle \circ [r, [t]] & \equiv p \circ r \rightarrow t; \dots ; \\
p \circ f^n \circ r & \rightarrow /w \circ [h \circ f^{n-1} \circ r, \dots, h \circ f \circ r, h \circ r, t]; \dots \\
& \text{since } /w \circ [t] \equiv t
\end{aligned}$$

when $g=w$ lemma 4 gives

$$\begin{aligned}
\text{Gen}: \langle p, f, w \circ [h \circ s1, s2] \rangle \circ [r, t] & \equiv p \circ r \rightarrow t; \dots ; \\
p \circ f^n \circ r & \rightarrow /w \circ [h \circ f^{n-1} \circ r, \dots, h \circ f \circ r, h \circ r, t]; \dots
\end{aligned}$$

Thus both sides of the theorem have identical expansions, therefore they denote the same function. $\square$