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NOTES ON APPLICATION OF COMPLETE POSETS TO RED LANGUAGES

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These notes include a lot of unattributed folklore noticed independently by many people. General properties of complete posets used here should not be credited to BKR

$\Rightarrow Fa \le_Y Fb$

Complete posets, ~~[illegible]~~ cartesian products, monotonicity are as in Sec. 6 of RC 4713. [Sec. Defined Data Types, Part II]

A map $F: X \to Y$ (where X, Y are complete posets) is continuous iff every nonempty linearly ordered subset L of X has FL ~~[illegible]~~ with a lub ~~[illegible]~~ in Y and ~~[illegible]~~

$$\mathrm{lub}_Y\, FL = F\, \mathrm{lub}_X\, L.$$

In places like Manna, Ness, & Vuillemin, only sets of form $L = \{a_i \mid i = 0, 1, \ldots\}$ are considered, where $a_0 \le a_1 \le a_2 \le \ldots$ in X. When completeness and continuity are limited to this kind of linearly ordered subset, we speak of ω-completeness and ω-continuity. In general, the prefix [ω-] could be stuck onto almost everything we say.

Continuity implies monotonicity. [Remark: We really should say isotonic, not monotonic, to agree with lattice theory.]

$\Rightarrow \mathrm{lub}_Y F\{x, x'\} = F\, \mathrm{lub}_X \{x, x'\}$

$\Rightarrow \mathrm{lub}_Y \{Fx, Fx'\} = Fx'$

$\Rightarrow Fx \le Fx'$.

The set of all continuous $F : X \to Y$ becomes a complete poset $Z = [X \to Y]$ when we agree that

$$F \leq G \text{ in } Z \text{ iff}$$
$$Fx \leq Gx \text{ in } Y, \text{ all } x \text{ in } X;$$

$$\perp_Z \text{ has } \perp_Z x = \perp_Y, \text{ all } x \text{ in } X.$$

See "[illegible] Note"

The evaluation map

$$\underset{\sim}{E} : [X \to Y] \times X \to Y$$

with

$$\underset{\sim}{E}(F, x) = Fx$$

is continuous.

Another way to get continuous maps is by composition: if $F : X \to Y$ and $G : Y \to Z$ are continuous, then $G \circ F : X \to Z$ is continuous.

Let D_α be a complete poset for each α in some set I. A map

$$F : \mathop{\times}_{\alpha \text{ in } I} D_\alpha \longrightarrow Y$$

is continuous iff it is continuous in each variable separately. To be continuous in D_α separately means this: for each choice of elements x_β in D_β for all $\beta \neq \alpha$, the map

$$x \text{ in } D_\alpha \longmapsto \boxed{\begin{array}{c}\underline{x} = \\ (\lambda \gamma \text{ in } I)\left(\underline{\text{if}}\ \gamma = \alpha\ \underline{\text{then}}\ x\ \underline{\text{else}}\ x_\gamma\right)\end{array}} \text{ in } \mathop{\times}_{\alpha \text{ in } I} D_\alpha$$

$$\longmapsto F\underline{x} \text{ in } Y$$

is continuous.

EXAMPLE: We have $F : D_1 \times D_2 \longrightarrow Y$ and want to show F is continuous. For each x_2 in D_2 the map F_1 with

$$x \text{ in } D_1 \longmapsto (x, x_2) \longmapsto F(x, x_2) \text{ in } Y$$

may be continuous. Also, for each x_1 in D_1, the ~~~~ map F_2 with

$$x \text{ in } D_2 \longmapsto (x_1, x) \longmapsto F(x_1, x) \text{ in } Y$$

may be continuous. If F_1 is indeed continuous for each choice of x_2 and F_2 is indeed continuous for each choice of x_1, then F is continuous.

Continuity can also be obtained for complete posets with special properties. A member y of D is <u>finite</u> iff every linearly ordered subset L of D with lub $L = y$ has y in L. If every member of D is finite we say D is <u>everywhere</u> <u>finite</u>.

EXAMPLE: Consider partial functions mapping $NN = \{0, 1, 2, \ldots\}$ into NN, with inclusion

An important complete poset is the truth value poset

$$TV = \begin{tikzcd}[column sep=small] tt & & ff \\ & uu \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

with bottom uu. Given a complete poset $(D, \leq, \perp)$, we have the <u>conditional</u> map

$$\underline{\text{cond}} : TV \times D \times D \longrightarrow D$$

defined by

$$\underline{\text{cond}}(tt, x, y) = x;$$
$$\underline{\text{cond}}(ff, x, y) = y;$$
$$\underline{\text{cond}}(uu, x, y) = \perp.$$

We can prove continuity by showing continuity in each variable separately. For x and y this is trivial: when we choose fixed p and y, for example, the map $x \longmapsto \underline{\text{cond}}\,(p, x, y)$ is either the identity map $\underset{\sim}{1}_D$ or is a constant.

We need continuity of $\underline{\text{cond}}\,(p, x, y)$ in p when x, y are held fixed. By $\bot \leq x$ and $\bot \leq y$, we do have monotonicity. In general monotonicity would not imply continuity, but TV is so simple that monotonicity does imply continuity. The nonempty linearly ordered subsets of TV are just

$$\{uu\} \quad \{tt\} \quad \{ff\}$$
$$\{uu, tt\} \quad \{uu, ff\}$$

and each one contains its own least upper bound. In general, a complete poset such that every ~~nonemptly~~ nonempty linearly ordered subset contains its own lub is said to be *discrete*.

Given any set E we can get a discrete complete poset easily. Let $\perp$ be anything not in E and let D be $E \cup \{\perp\}$. Let $a \leq b$ iff $a = \perp$ or $a = b$ for all a, b in D. Then $(D, \leq, \perp)$ is a discrete poset. The truth value poset is derived from the usual set $\{tt, ff\}$ of ~~the~~ truth values in this way.

Another way to get a discrete poset is to take a cartesian product of finitely many discrete posets.

Whenever D, Y are complete posets and D is discrete, any monotonic map from D into Y is continuous.

Initial algebras have often been considered in computer science, usually under obscure names like "abstract syntax" or "constructor syntax." For RED languages we are concerned with an initial ~~alg~~ algebra whose carrier is the set E of "expressions" and whose operators include a distinguished dyadic operator ap. Members of E that can be generated without ap form the set C of "constants" and members of E generated by niladic operators form the set A of "atoms."

Adding $\perp$ to ~~E~~ E gives us a discrete (complete) poset. The operators can be extended to allow arguments from $D = E \cup \{\perp\}$ by ~~the~~ letting

$$(k[e_1, \ldots, e_n] \text{ in } D) = \begin{cases} k[e_1, \ldots, e_n] \text{ in } E & \text{if all } e_i \text{ are in } E \\ \perp & \text{if some } e_i \text{ are } \perp \end{cases}$$

Then each operator k is monotonic from D^n into D. and ~~is~~ hence ~~continuous~~ continuous, since D^n is discrete.

Similarly we extend the representation function ρ from C to maps from C to E, so that ρ is now in $[D \to [D \to D]]$ and $(\rho c)d = \perp$ if c, d are not both in C.

Given F, G in $[D \to D]$ we can define a new map $T(F,G)$ from D into D:

$$T(F,G)x = \begin{cases} \perp & \text{if } x = \perp \\ k(Fy_1, \ldots, Fy_n) & \text{if some } k \neq \underline{ap} \text{ has} \\ & \quad x = k(y_1, \ldots, y_n) \neq \perp \\ G((\rho Fy)Fz) & \text{if} \\ & \quad x = \underline{ap}(y,z) \neq \perp \end{cases}$$

(The case x in A is not overlooked here. It is carried along with no fuss by consistently allowing for niladic choices of k.)

Since $T(F,G)\perp = \perp \leq T(F,G)x$ for all x, $T(F,G)$ is monotonic and hence in $[D \to D]$. Thus we have

$$T : ([D \to D] \times [D \to D]) \to [D \to D]$$

Now consider the basic properties desired for a meaning function $\mu : E \to C$ (partial):

$$\mu\, k(e_1, \ldots, e_n) = k(\mu e_1, \ldots, \mu e_n)$$
if $k \neq \underline{ap}$ & $e_1, \ldots, e_n$ are in E;

$$\mu\, \underline{ap}\,(e,f) = \mu((\rho \mu e)\, \mu f)$$
if e, f are in E.

Extending ~~to~~ μ so as to be in $[D \to D]$ by letting $\mu x = \perp$ whenever x is not in the old domain of μ, we see that

$$\mu = T(\mu, \mu) \text{ iff}$$
μ has the desired properties.

For any F in $[D \to D]$ we have $T(F, F)$ in $[D \to D]$ and we want to show that $F \mapsto T(F,F)$ has at least one fixpoint and so the language has at least one meaning function.

Let $(T\delta)F$ be $T(F,F)$, so that $(T\delta) : [D \to D] \to [D \to D]$. We want to show that $(T\delta)$ is continuous so as to get a fixpoint

$$\mu = \text{lub}\ \{ (T\delta)^m \perp \mid m = 0, 1, 2, \ldots \}$$

where $\perp$ is the bottom of $[D \to D]$. Since $(T\delta)$ is a composition

$$F \overset{}{\underset{\delta}{\longmapsto}} (F,F) \underset{T}{\longmapsto} T(F,F)$$

and δ is clearly continuous, it will suffice to show T is continuous.

For any G we show that T_1^G is continuous, where $T_1^G F = T(F, G)$. For any ~~K~~ nonempty linearly ordered subset L of $[D \to D]$, we need

$$\text{lub } T_1^G L \text{ exists } \& \text{ lub } T_1^G L = T_1^G \text{ lub } L.$$

Thus we need, for all x in D,

$$\text{lub}_D (T_1^G L) x \text{ exists } \& \text{ lub}_D (T_1^G L) x = (T_1^G \text{ lub} L) x,$$

where $(T_1^G L) x = \{ (T_1^G F) x \mid F \text{ in } L \}$.
In computing $(T_1^G F) x$ we use the same case in the def. of $T(F, G)$ for all choices of ~~x~~ F: only x determines which case to use. For example, if $x = \underline{ap}(y, z) \neq \bot$, then

$$(T_1^G L) x = \{ G((\rho F y) F z) \mid F \text{ in } L \}$$

Since $F_1 \leq F_2$ in L implies $G((\rho F_1 y) F_1 z) \leq G((\rho F_2 y) F_2 z)$ by monotonicity, $\text{lub}_D (T_1^G L) x$ does exist in this case. Furthermore,

$\mathrm{lub}_D(T_1^G L)\,x =$

$= \mathrm{lub}_D\{\, G((\rho F y) F z) \mid F \text{ in } L \,\} =$

$= G((\rho\ \mathrm{lub}_D\{Fy \mid F \text{ in } L\})\ \mathrm{lub}_D\{Fz \mid z \text{ in } L\})$

by continuity of G, ρ, and $\underset{\sim}{E}$ of page 2

$= G((\rho\ (\mathrm{lub}\,L)y)\ (\mathrm{lub}\,L)z)$

by def. of lub in $[D \to D]$

$= (T_1^G\ \mathrm{lub}\,L)\,x$ [as wanted]

because $T(\mathrm{lub}\,L, G)\,x$ uses the same case in the def. of T as $T(F,G)\,x$ for any F in L.

The other cases for x are treated similarly.

For any F we show that $T_2^F E$ is ~~cont~~ continuous, where $T_2^F G = T(F,G)$. This is like the proof for T_1^G but is even easier.

Since T is continuous in both variables separately, it is continuous. Therefore $(T\delta)$ is continuous and does have the desired fixpoint.

We now have a meaning function μ in $[D \to D]$ (or $\mu : E \to E$ partial, if we ignore $\perp$ values) with the basic properties wanted. The explicit equation for μ on page 11 can be used to derive further properties.

EXAMPLE: For any e in E, μe is in $C \cup \{\perp\}$. Since $\mu e = [(T\delta)^m \perp] e$ for some choice of m (by discreteness of D and the def. of lub's in $[D \to D]$), it suffices to show

$$\left[((T\delta)^m \perp) x \text{ in } C \cup \{\perp\}, \text{all } x \text{ in } D \right]$$

for all m. This follows from the def. of T by induction on m.

EXAMPLE: $C = \{c \mid \mu c = c\}$ follows from the previous example and $\mu c = c$ for all c in C. Likewise $\mu\mu = \mu$. (Use structural induction to show $\mu c = c$ for all c in C.)

With a bit more work we could avoid the ~~discreteness assumption~~ use of discreteness of D, so that situations where E is embedded in a more complex complete poset with ~~more~~ more complex extensions of the operators and of ρ could be considered.

~~Now~~ Why might more complex posets be of interest? Consider an atom HEAD such that

$$(\rho\ \text{HEAD})d = x$$

whenever d has the form $k(x, \ldots)$ for $k \neq \underline{ap}$ and k not niladic. Thus

ρHEAD extracts the head of the argument list to the operator used in building the argument to ρHEAD. (If d is an atom then $(\rho HEAD)\, d = \perp$ could be specified, but $(\rho\, HEAD) d =$ ERROR_IN_HEAD for a special atom ERROR_IN_HEAD would be much better.) We can ~~too~~ handle ρ as above of course. If z is an expression such that $\mu z = \perp$, we will ~~to~~ get

$$\mu \; \underline{ap}\,(HEAD,\ k\cdot(53, z)) = \perp .$$

We could however imagine a map μ_* which satisfies, among other things, the equation

$$\mu_* \; \underline{ap}\,(HEAD,\ k(53, z)) = 53$$

for all z, even for $\mu_* z = \perp$. Can we got μ_* to act very much like μ otherwise? Can we get μ_* in any natural way, as the least fixpoint of a functional with a definition like that of $T(F, G)$?

☆ MORE TO COME ☆

APPENDIX: Fixpoint Theorem for $T: X \to X$, continuous.
The set $\{T^n \perp \mid n = 0, 1, 2, \ldots\} \subseteq X$ is nonempty and linearly ordered: $\perp \leq T\perp$ leads to $T^n\perp \leq T^{n+1}\perp$ for all n by monotonicity.
Let $\varphi = \text{lub}\,\{T^n\perp \mid n = 0, 1, 2, \ldots\}$ in the complete poset X. By continuity of T,

$$
\begin{aligned}
T\varphi &= \text{lub}\,\{T(T^n\perp) \mid n = 0, 1, 2, \ldots\} \\
&= \text{lub}\,\{T^p\perp \mid p = 1, 2, \ldots\} \\
&= \text{lub}\,(\{T^p\perp \mid p = 1, 2, \ldots\} \cup \{\perp\}) \\
&= \varphi
\end{aligned}
$$

[Argument first given for complete lattices by Scott, with inspiration from Tarski and from Kleene.]