

Item #198

# SOME BASIC FL LAWS

Ed Wimmers

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## Comments:

- 1) Recall that all variables represent expressions in normal form and are not members of the collapsing set.
- 2) In order to prove the Laws, we use the Axioms and facts about normal forms.  
The axioms themselves are sufficiently rich to handle the case that different functions produce different errors.  
The axioms also handle I/O correctly.  
In order to satisfy axioms 41, 44, and 74, it is probably necessary that  $AA:f:x = AA:g:x$  for all non-sequence values  $x$  even if  $f$  and  $g$  are different functions.
- 3) To enhance understanding, the following definitions are given.  
Since only the axioms are needed, these definitions should be viewed as comments that aid in understanding the axioms.  
Def dom == K:sl @ [ K:id , id ]  
Def DEL == (id -> K:id;K:X) where X is something like nt or mkerr.
- 4) To enhance readability, the axioms are informally divided as follows:  
1-19) These axioms are sort of like definitions.  
20-39) These are trivial axioms.  
40-59) These are substantial axioms.  
60-79) These axioms deal with the functions gamma and dom.  
80-99) These are axioms that need a gamma qualification.

```
Axm 1 C:f:x @ g == f @ [ K:x , g ]
Axm 2 comp == C : ( \:app @ ar ) @ dom:comp
Axm 3 cons == C : ( AA:app @ distr ) @ dom:cons
Axm 4 id @ f == f
Axm 5 K == C:sl
Axm 6 MakeFunc == C:app
```

```
Axm 21 AA == AA @ MakeFunc
Axm 22 C:f == C:(f@DEL:ispair)
Axm 23 DEL:ispair == [sl,s2] @ DEL:ispair
Axm 24 ( f @ g ) : x == f : ( g : x )
Axm 25 f @ g == comp : < f , g >
Axm 26 cons : < f1 , ... , fn > == [ f1 , ... , fn ]
Axm 27 AA:f:<x1,...,xn> == <f:x1,...,f:xn>
Axm 28 distr == AA:(DEL:ispair) @ distr
Axm 29 dom:[f@sl,g@s2] @ DEL:ispair == [dom:f@sl,dom:g@s2] @ DEL:ispair
Axm 30 AA:sl @ distr == sl @ dom:distr
Axm 31 f @ AA:g == AA:g if f = dom:cons, dom:comp, DEL:isseq
Axm 32 [sl,...,si,g@s2]@[f1,...,fj] == [f1,...,fi,g@fj] if j=i+1
Axm 33 C:f:x:y == f:<x,y>
Axm 34 si:<x1,...,xj> == xi if i <= j
Axm 35 f @ ( g @ h ) == ( f @ g ) @ h == f @ g @ h
Axm 36 dom @ DEL == DEL
```

```
Axm 41 comp @ AA:(AA@gamma) == AA @ comp @ AA@gamma
Axm 42 app @ [ C:f @ g , h ] == f @ [ g , h ]
```

```

Axm 43  app @ [ K:f , g ] == f @ g
Axm 44  C : ( AA:f @ dist1 @ [s1,AA:id@s2] ) == AA @ C:f

Axm 61  gamma : f == f      if f is a primitive other than IO or app
Axm 62  gamma : app @ [s1,s2] == app @ [ gamma @ s1 , s2 ]
Axm 63  gamma @ AA == AA @ gamma
Axm 64  gamma @ C:f == C:(gamma:f)
Axm 65  gamma @ comp == comp @ AA:gamma @ dom:comp
Axm 66  gamma @ cons == cons @ AA:gamma @ dom:cons
Axm 67  gamma @ gamma == gamma
Axm 68  gamma @ DEL == DEL @ gamma
Axm 69  dom : f == id      if f = K, id
Axm 70  dom : ( K : x ) == id
Axm 71  dom:[f1,...,fn] == dom: fn @ ... @ dom:f1
Axm 72  dom @ dom == dom
Axm 73  gamma @ dom == dom @ gamma
Axm 74  dom @ AA == AA @ dom
Axm 75  dom:distr @ [ AA:id @ f , g ] == [ AA:id @ f , g ]

```

In the following Axioms, all the g's need to satisfy the property that  
 $\text{gamma} : g == g$ . (The f's need not satisfy the property.)

```

Axm 81  AA:f @ AA:g2 == AA:(f@g2) @ AA:(dom:g2)
Axm 82  AA:app@AA:[gamma@g1,g2] ==
        AA:(app@[gamma@g1,g2])@dom:(AA:[gamma@g1,g2])
Axm 83  C:f @ g == C:(f@[g@s1,s2]) @ dom:g
Axm 84  distr@[[g1,...,gn],g] == [[g1,g],...,[gn,g]]@dom:[[g1,...,gn],g]
Axm 85  AA:f @ [ g1 , ... , gn ] == [ f@g1 , ... , f@gn ] @ dom:[g1,...,gn]
Axm 86  [ f1@g , f2 , ... , fn ] == [ f1@g , f2 , ... , fn ] @ dom:g
Axm 87  dom:g @ dom:g == dom:g
Axm 88  distr@[AA:g1@s1,g2@s2] ==
        AA:[g1@s1,g2@s2]@distr@dom:[AA:g1@s1,g2@s2]

```

Law 10  $\text{gamma} @ K == K$

Proof:

```

gamma @ K
== gamma @ C:s1      A5
== C:(gamma:s1)      A64
== C:s1              A61
== K                  A5

```

Law 20  $C:(f@distr) @ AA:g == C:(f@AA:[g@s1,s2]@distr) @ AA:(dom:g)$   
if  $\text{gamma}:g == g$

Proof:

```

C:(f@distr)@AA:g
== C:(f@distr@[AA:g@s1,s2]) @ dom:(AA:g)      A83
== C:(f@AA:[g@s1,s2]@distr@dom:[AA:g@s1,s2]) @ dom:(AA:g)      A88
== C:(f@AA:[g@s1,s2]@distr@dom:[AA:g@s1,s2]@DEL:ispair)@dom:(AA:g)      A22
== C:(f@AA:[g@s1,s2]@distr@[dom:(AA:g)@s1,s2]@DEL:ispair)@dom:(AA:g)      A29
== C:(f@AA:[g@s1,s2]@distr@[dom:(AA:g)@s1,s2])@dom:(AA:g)      A22
== C:(f@AA:[g@s1,s2]@distr@[dom:(AA:g)@s1,s2])@dom:(dom:(AA:g))      A72
== C:(f@AA:[g@s1,s2]@distr) @ dom:(AA:g)      A83
== C:(f@AA:[g@s1,s2]@distr) @ AA:(dom:g)      A74

```

Law 30  $K : x : y == x$

Proof:

|                                 |     |
|---------------------------------|-----|
| $K : x : y$                     | A24 |
| $== C : s1 : x : y$             | A5  |
| $== s1 : \langle x : y \rangle$ | A33 |
| $== x$                          | A34 |

Law 35  $id : x == x$

Proof:

|                       |     |
|-----------------------|-----|
| $id : x$              |     |
| $== id : (K : x : x)$ | L30 |
| $== (id @ K : x) : x$ | A24 |
| $== K : x : x$        | A4  |
| $== x$                | L30 |

Law 40  $gamma : (f@g) == gamma:f @ gamma:g$

Proof:

|                                                                 |         |
|-----------------------------------------------------------------|---------|
| $gamma : (f@g)$                                                 |         |
| $== (gamma@comp) : \langle f,g \rangle$                         | A24,A25 |
| $== (comp@AA:gamma@dom:comp) : \langle f,g \rangle$             | A65     |
| $== comp:(AA:gamma:(dom:comp:(AA:id:\langle f,g \rangle)))$     | L35,A27 |
| $== (comp @ AA:gamma @ dom:comp @ AA:id) : \langle f,g \rangle$ | A24     |
| $== (comp @ AA:gamma @ AA:id) : \langle f,g \rangle$            | A31     |
| $== (comp @ AA:gamma) : \langle f,g \rangle$                    | A27,L35 |
| $== comp : \langle gamma:f, gamma:g \rangle$                    | A27     |
| $== gamma : f @ gamma : g$                                      | A25     |

Law 50  $gamma : [ g1 , \dots , gn ] == [ gamma:g1 , \dots , gamma:gn ]$

Proof:

|                                                                       |         |
|-----------------------------------------------------------------------|---------|
| $gamma : [ g1 , \dots , gn ]$                                         |         |
| $== (gamma@cons) : \langle g1, \dots, gn \rangle$                     | A26,A24 |
| $== (cons@AA:gamma@dom:cons) : \langle g1, \dots, gn \rangle$         | A66     |
| $== cons:(AA:gamma:(dom:cons:\langle g1, \dots, gn \rangle))$         | A24     |
| $== cons:(AA:gamma:(dom:cons:(AA:id:\langle g1, \dots, gn \rangle)))$ | L35,A27 |
| $== cons:(AA:gamma:((dom:cons@AA:id):\langle g1, \dots, gn \rangle))$ | A24     |
| $== cons:(AA:gamma:(AA:id:\langle g1, \dots, gn \rangle))$            | A31     |
| $== cons:(AA:gamma:\langle g1, \dots, gn \rangle)$                    | L35,A27 |
| $== cons:(\langle gamma:g1, \dots, gamma:gn \rangle)$                 | A27     |
| $== [ gamma:g1 , \dots , gamma:gn ]$                                  | A26     |

Law 60  $AA == C : ( AA:app @ dist1 @ [s1,AA:id@s2] )$

Proof:

|                                             |     |
|---------------------------------------------|-----|
| $AA$                                        |     |
| $== AA @ MakeFunc$                          | A21 |
| $== AA @ C:app$                             | A6  |
| $== C : ( AA:app @ dist1 @ [s1,AA:id@s2] )$ | A44 |

Law 70  $app @ [ cons @ f , g ] == AA:app @ distr @ [ dom:cons @ f , g ]$

Proof:

$app @ [ cons @ f , g ]$

== app @ [ C:(AA:app@distr) @ dom:cons @ f , g ] A3  
 == (AA:app@distr) @ [ dom:cons @ f , g ] A42

Law 80 app @ [ AA @ f , g ] == AA:app @ distl @ [ f , AA:id @ g ]

Proof:

app @ [ AA @ f , g ]  
 == app @ [ C:(AA:app@distl@[s1,AA:id@s2]) @ f , g ] L60  
 == AA:app @ distl @ [s1,AA:id@s2] @ [ f , g ] A42  
 == AA:app @ distl @ [ f , AA:id @ g ] A32

Law 90 AA:f @ distr == AA:(f@DEL:ispair) @ distr

Proof:

AA:f @ distr  
 == AA:f @ AA:(DEL:ispair) @ distr A28  
 == AA:f @ dom:(AA:(DEL:ispair)) @ distr A36,A74  
 == AA:f @ dom:(AA:(DEL:ispair)) @ dom:(AA:(DEL:ispair)) @ distr A87  
 == AA:f @ AA:(DEL:ispair) @ AA:(dom:(DEL:ispair)) @ distr A36,A74  
 == AA:(f@DEL:ispair) @ AA:(dom:(DEL:ispair)) @ distr A81  
 == AA:(f@DEL:ispair) @ AA:(DEL:ispair) @ distr A36  
 == AA:(f@DEL:ispair) @ distr A28

Law 100 cons @ AA:K == K @ AA:id

Proof:

First note the following fact:

Fact 1) dom:(AA:[gamma@K@s1,s2]) @ distr == distr

Proof:

dom:(AA:[gamma@K@s1,s2]) @ distr  
 == dom:(AA:[K@s1,s2]) @ distr L10  
 == AA:(dom:[K@s1,s2]) @ distr A74  
 == AA:(dom:[K@s1,s2]@DEL:ispair) @ distr L90  
 == AA:(dom:[K@s1,id@s2]@DEL:ispair) @ distr A4  
 == AA:([dom:K@s1,dm:id@s2]@DEL:ispair) @ distr A29  
 == AA:([id@s1,id@s2]@DEL:ispair) @ distr A69  
 == AA:([s1,s2]@DEL:ispair) @ distr A4  
 == AA:(DEL:ispair) @ distr A23  
 == distr A28

cons @ AA:K

== C:(AA:app @ distr) @ dom:cons @ AA:K A3  
 == C:(AA:app @ distr) @ AA:K L31  
 == C:(AA:app @ AA:[K@s1,s2] @ distr) @ AA:(dom:K) L20  
 == C:(AA:app @ AA:[K@s1,s2] @ distr) @ AA:id A69  
 == C:(AA:app @ AA:[gamma@K@s1,s2] @ distr) @ AA:id L10  
 == C:(AA:(app@[gamma@K@s1,s2])@dom:(AA:[gamma@K@s1,s2])@distr)@AA:id A82  
 == C:(AA:(app@[gamma@K@s1,s2])@distr) @ AA:id F1  
 == C:(AA:(app@[K@s1,s2])@distr) @ AA:id L10  
 == C:(AA:(sl@[s1,s2])@distr) @ AA:id A43  
 == C:(AA:(sl@[s1,s2]@DEL:ispair)@distr) @ AA:id L90  
 == C:(AA:(sl@DEL:ispair)@distr) @ AA:id A23  
 == C:(AA:sl@distr) @ AA:id L90  
 == C:(sl@dom:distr) @ AA:id A30  
 == C:(sl @ dom:distr @ [ AA:id @ s1 , s2 ] ) @ dom:(AA:id) A83  
 == C:(sl @ [ AA:id @ s1 , s2 ] ) @ dom:(AA:id) A75

== C:( sl ) @ AA:id  
 == K @ AA:id

A83  
 A5

Law 110  $K : \langle x_1, \dots, x_n \rangle == [ K : x_1, \dots, K : x_n ]$

Proof:

$K : \langle x_1, \dots, x_n \rangle$   
 ==  $K : ( AA:id : \langle x_1, \dots, x_n \rangle )$  L35,A27  
 ==  $( K @ AA:id ) : \langle x_1, \dots, x_n \rangle$  A24  
 ==  $( cons @ AA:K ) : \langle x_1, \dots, x_n \rangle$  L100  
 ==  $cons : ( AA:K : \langle x_1, \dots, x_n \rangle )$  A24  
 ==  $cons : \langle K : x_1, \dots, K : x_n \rangle$  A27  
 ==  $[ K : x_1, \dots, K : x_n ]$  A26

Law 120  $[ f_1, \dots, f_n ] @ g == [ f_1@g, \dots, f_n@g ]$  if  $gamma:g == g$

Proof:

First note the following facts:

Fact 1)  $dom:[ [ K : f_1, \dots, K : f_n ], g ] == dom:g$

Proof:

$dom:[ [ K : f_1, \dots, K : f_n ], g ]$   
 ==  $dom:g @ dom:[ K : f_1, \dots, K : f_n ]$  A71  
 ==  $dom:g @ dom:( K : f_n ) @ \dots @ dom:( K : f_1 )$  A71  
 ==  $dom:g @ id @ \dots @ id$  A70  
 ==  $dom:g$  A4

Fact 2)  $dom:[ [K: f_1,g], \dots, [K: f_n,g] ] == dom:g$

Proof:

$dom:[ [K: f_1,g], \dots, [K: f_n,g] ]$   
 ==  $dom:[K: f_n,g] @ \dots @ dom:[K: f_1,g]$  A71  
 ==  $dom:g @ dom:(K: f_n) @ \dots @ dom:g @ dom:(K: f_1)$  A71  
 ==  $dom:g @ id @ \dots @ dom:g @ id$  A70  
 ==  $dom:g @ \dots @ dom:g$  A4  
 ==  $dom:g$  A87

$[ f_1, \dots, f_n ] @ g$

==  $cons:\langle f_1, \dots, f_n \rangle @ g$  A26  
 ==  $( C:(AA:app@distr) @ dom:cons ) : \langle f_1, \dots, f_n \rangle @ g$  A3  
 ==  $( C:(AA:app@distr) @ dom:cons ) : (AA:id:\langle f_1, \dots, f_n \rangle) @ g$  A27,L35  
 ==  $( C:(AA:app@distr) @ dom:cons @ AA:id ) : \langle f_1, \dots, f_n \rangle @ g$  A24  
 ==  $C:(AA:app@distr) : ( AA:id : \langle f_1, \dots, f_n \rangle ) @ g$  A24  
 ==  $C:(AA:app@distr) : \langle f_1, \dots, f_n \rangle @ g$  A27,L35  
 ==  $(AA:app@distr) @ [ K : \langle f_1, \dots, f_n \rangle , g ]$  A1  
 ==  $(AA:app@distr) @ [ [ K : f_1, \dots, K : f_n ], g ]$  L110  
 ==  $AA:app @ [ [ K : f_1, g ], \dots, [ K : f_n, g ] ] @ dom:g$  A84,F1  
 ==  $[ app@[K: f_1,g], \dots, app@[K: f_n,g] ] @ dom:g @ dom:g$  A85,F2  
 ==  $[ app@[K: f_1,g], \dots, app@[K: f_n,g] ] @ dom:g$  A87  
 ==  $[ f_1@g, \dots, f_n@g ] @ dom:g$  A43  
 ==  $[ f_1@g, \dots, f_n@g ]$  A86

Comment: It is interesting to note that Axiom 42 is used (sometimes indirectly) in the proof of so many of the laws.