

Church-Rosser FACT SHEET

NOTATION

LET $\rightarrow$ BE A RELATION ON DOMAIN D . WE WRITE $x \rightarrow y$ IF $(x, y) \in \rightarrow$.

$\xrightarrow{+}$ IS THE TRANSITIVE CLOSURE OF $\rightarrow$

$\xrightarrow{*}$ IS THE REFLEXIVE-TRANSITIVE CLOSURE OF $\rightarrow$

$\longleftrightarrow$ IS THE SYMMETRIC CLOSURE OF $\rightarrow$

$$x \downarrow y \iff \exists z \text{ s.t. } x \xrightarrow{*} z \wedge y \xrightarrow{*} z$$

$$x \uparrow y \iff \exists z \text{ s.t. } z \xrightarrow{*} x \wedge z \xrightarrow{*} y$$

DEFINITIONS

- $\rightarrow$ IS NOETHERIAN IF THERE IS NO INFINITE SEQUENCE $x_1 \rightarrow x_2 \rightarrow \dots$
- $\rightarrow_\alpha$ IS THE ORDERING OF $\rightarrow$ ON DOMAIN D_α WHERE:
 - D_α IS THE SET OF EQUIVALENCE CLASSES OF ELEMENTS OF D WRT THE EQUIVALENCE RELATION: $a \equiv b \iff a \xrightarrow{*} b \wedge b \xrightarrow{*} a$
 - $a \rightarrow_\alpha b \iff \exists a' \in a, b' \in b \text{ s.t. } a' \rightarrow b'$
- $\rightarrow$ IS CR $\iff x \longleftrightarrow y \Rightarrow x \downarrow y$
- $\rightarrow$ IS CONFLUENT $\iff \forall xy \quad x \uparrow y \Rightarrow x \downarrow y$
- $\rightarrow$ IS LOCALLY CONFLUENT $\iff \forall xyz \quad x \rightarrow y \wedge x \rightarrow z \Rightarrow y \downarrow z$
- $\rightarrow$ IS STRONGLY CONFLUENT $\iff \forall xyz \quad x \rightarrow y \wedge x \rightarrow z \Rightarrow \exists u \quad y \xrightarrow{*} u \wedge (z = u \vee z \rightarrow u)$

CHURCH-ROSSER FACT SHEET (CONT.)

FACTS

- $\rightarrow$ IS CR $\iff \rightarrow$ IS CONFLUENT
- $\rightarrow$ IS STRONGLY CONFLUENT $\implies \rightarrow$ IS CONFLUENT
- LET $\rightarrow$ BE NOETHERIAN. THEN
 $\rightarrow$ IS LOCALLY CONFLUENT $\iff \rightarrow$ IS CONFLUENT
- $\rightarrow_c$ IS CONFLUENT $\& \rightarrow_c^* = \rightarrow^* \implies \rightarrow$ IS CONFLUENT
- $\rightarrow$ IS CONFLUENT $\iff \rightarrow_\alpha$ IS CONFLUENT

NOTE: THERE ARE MANY OTHER SUFFICIENT CONDITIONS FOR $\rightarrow$ TO BE CONFLUENT, BUT NO NON-TRIVIAL NECESSARY & SUFFICIENT CONDITIONS ARE KNOWN.

TERM REWRITING SYSTEMS

LET F BE A SET OF FUNCTION SYMBOLS.

LET V " " " " VARIABLES.

LET N_+^* BE THE SET OF ALL SEQUENCES OF POSITIVE NUMBER
(Λ IS THE EMPTY SEQUENCE.)

A TERM IS A TREE WHERE INTERNAL NODES ARE
ELEMENTS OF F AND LEAVES ARE ELEMENTS OF $F \cup V$.

LET T BE A TERM.

DEFINITIONS

$V(T)$ IS THE SET OF VARIABLES IN T

THE SUBTERM OF T AT $u \in N_+^*$, WRITTEN T/u , IS:

T IF $u = \Lambda$

T_i/p IF $T = \begin{array}{c} f \\ \swarrow \quad \searrow \\ T_1 \dots T_i \dots T_n \end{array}$ & $u = ip$

A SUBSTITUTION θ IS A FUNCTION FROM VARIABLES TO TERMS.

$\theta(T)$, WHERE θ IS A SUBSTITUTION, IS THE RESULT OF REPLACING
EVERY VARIABLE x IN T WITH $\theta(x)$.

$T_1 :: T_2$ IS A (MOST-GENERAL) SUBSTITUTION WHICH UNIFIES
 T_1 AND T_2 . (IF SUCH A SUBSTITUTION EXISTS.)

A RULE IS A PAIR OF TERMS $\langle \alpha \rightarrow \beta \rangle$ WHERE $V(\beta) \subseteq V(\alpha)$

LET R BE A SET OF RULES.

T/U IS A REDEX IF $\langle \alpha \rightarrow \beta \rangle \in R$ & $T/U::\alpha$ EXISTS.

LET T/U BE A REDEX AS ABOVE. THE CONTRACTION OF T/U IS $\theta(\beta)$ WHERE $\theta = T/U::\alpha$.

WE NOW DEFINE A RELATION $\rightarrow$ ON TERMS:

$T \rightarrow T' \iff \exists u \in N_+^*, \langle \alpha \rightarrow \beta \rangle \in R$ S.T. $\theta = T/U::\alpha$ EXISTS
AND $T' = T[u \leftarrow \theta(\beta)]$

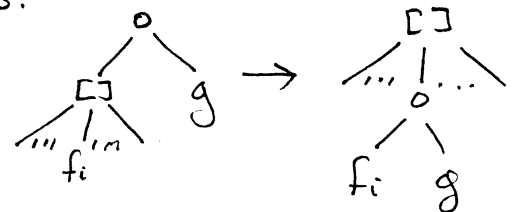
TRANSFORMATIONS

1. $[..., f_i, ...] \circ g \rightarrow [..., f_i \circ g, ...]$
2. $s_i \circ [..., f_i, ...] \rightarrow f_i$
3. $id \circ f \rightarrow f$
4. $f \circ id \rightarrow f$
5. $\alpha: f \circ [..., g_i, ...] \rightarrow [..., f \circ g_i, ...]$
6. $\alpha: f \circ \text{for}: \langle E, h \rangle \rightarrow \text{for}: \langle f \circ E, h \rangle$
7. $\text{for}: \langle E, h \rangle \circ f \rightarrow \text{for}: \langle E \circ [f \circ \text{Arg}, \text{Ind}], h \circ f \rangle$
8. $\text{len} \circ \text{for}: \langle E, g \rangle \rightarrow g$
9. $\text{rel} \circ [\text{for}: \langle E, f \rangle, g] \rightarrow E \circ [id, g]$
10. $\text{sel} \circ [g, \text{for}: \langle E, f \rangle] \rightarrow E \circ [id, g]$
11. $\text{ints to} \rightarrow \text{for}: \langle \text{Ind}, id \rangle$
12. $\text{distr} \rightarrow \text{for}: \langle [\text{rel} \circ [s_1 \circ \text{Arg}, \text{Ind}], s_2 \circ \text{Arg}], \text{len} \circ s_1 \rangle$
13. $\text{distl} \rightarrow \text{for}: \langle [s_1 \circ \text{Arg}, \text{rel} \circ [s_2 \circ \text{Arg}, \text{Ind}]], \text{len} \circ s_2 \rangle$
14. $\text{app} \circ [\text{cons} \circ f, g] \rightarrow \alpha: \text{app} \circ \text{distr} \circ [f, g]$
15. $\text{app} \circ [\alpha: f, g] \rightarrow \alpha: \text{app} \circ \text{distl} \circ [f, g]$
16. $\text{app} \circ [C: f \circ g, h] \rightarrow f \circ [g, h]$
17. $\text{len} \circ [f_1, ..., f_n] \rightarrow \sim_n$
18. $\text{Arg} \rightarrow s_1$
19. $\text{Ind} \rightarrow s_2$
20. $[s_1, s_2, ..., s_n] \rightarrow id$
21. $\sim f \circ g \rightarrow \sim f$

RULES

NOTE: THESE ARE REALLY PAIRS OF TREES; e.g., RULE 1

IS:



NOTE: COMPOSITION IS RESTRICTED TO BE BINARY.

THEOREM: THE REWRITING RELATION INDUCED ON FL PROGRAMS BY THIS SET OF RULES IS CR.

OUTLINE OF PROOF:

- DEFINE A RELATION $\twoheadrightarrow$ TO BE THE SIMULTANEOUS CONTRACTION OF DISJOINT REDEXES.

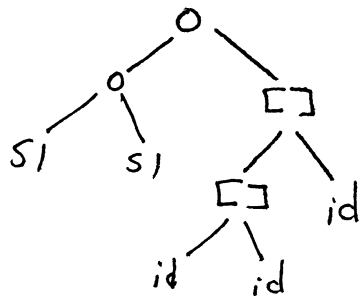
- LEMMA: $\twoheadrightarrow$ IS STRONGLY CONFLUENT; IN FACT:
 $P \twoheadrightarrow A \wedge P \twoheadrightarrow B \Rightarrow \exists C \text{ s.t. } A \twoheadrightarrow C \wedge B \twoheadrightarrow C$

PROOF: BY INDUCTION ON THE LENGTH OF P .

- $\xrightarrow{*} = \twoheadrightarrow^*$ AND $\twoheadrightarrow$ IS CONFLUENT $\Rightarrow \rightarrow$ IS CONFLUENT & CR

BUT

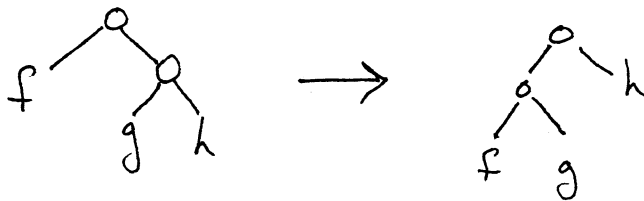
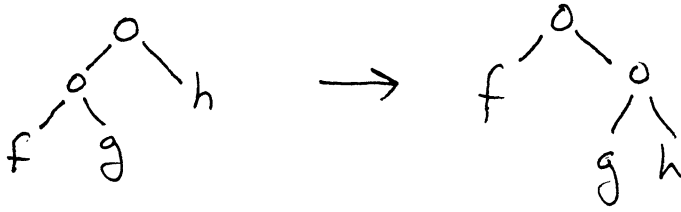
ALTHOUGH THE SYSTEM IS CR, IT DOESN'T HANDLE THE ASSOCIATIVITY OF COMPOSITION. CONSIDER:



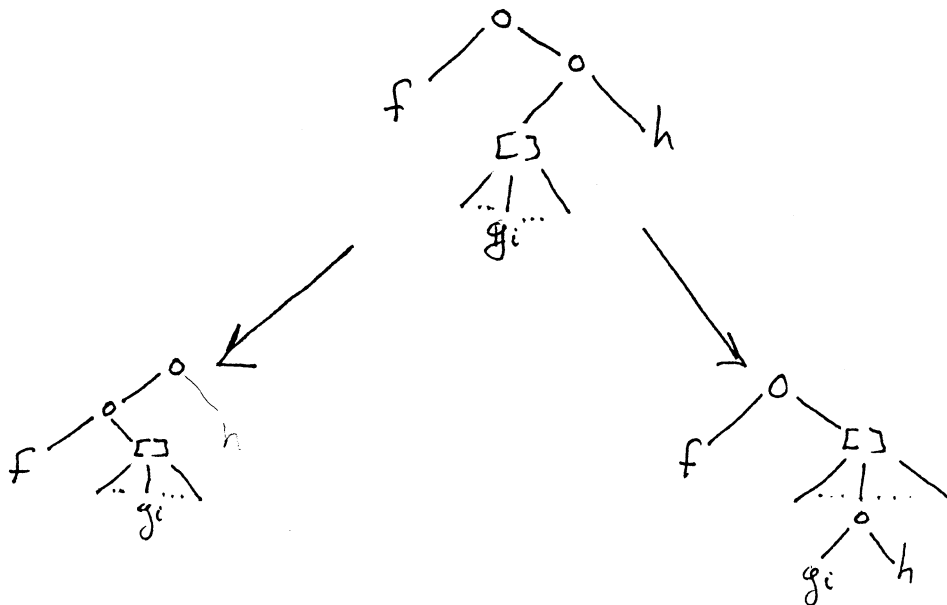
NO RULES APPLY.

ASSOCIATIVITY OF COMPOSITION

WE ADD TWO RULES:



UNFORTUNATELY, ADDING THESE RULES DISTURBS THE PROOF THAT $\rightarrow$ IS STRONGLY CONFLUENT. CONSIDER THE FOLLOWING:



(I have not been able to find a relation $\rightarrow_c$

THAT TAKES THE LEFT & RIGHT SIDES TO A COMMON TERM
IN ONE STEP, OTHER THAN $\rightarrow$. THIS REDUCES THE PROOF TO SHOWING
 $A \rightarrow B, A \rightarrow C \Rightarrow B \rightarrow D, C \rightarrow D$ - NO HELP AT ALL!)

IT IS STILL THE CASE THAT $\rightarrow$ IS LOCALLY CONFLUENT.

HOWEVER, $\rightarrow$ IS NOT NOETHERIAN, BECAUSE THE ASSOCIATIVE LAWS ARE INVERSES OF EACH OTHER. TO GET AROUND THIS,

WE FIRST SHOW THAT $\rightarrow_2$, THE ORDERING OF $\rightarrow$, IS NOETHERIAN. THEN $\rightarrow_2$ IS LOCALLY CONFLUENT BECAUSE $\rightarrow$ IS, SO $\rightarrow_2$ IS CONFLUENT. THEREFORE, $\rightarrow$ IS CONFLUENT.

NOTE: THE ASSOCIATIVE LAWS BY THEMSELVES FORM A CHURCH-ROSSER RELATION. HOWEVER, THE UNION OF TWO CR RELATIONS IS NOT NECESSARILY CR.

LEMMA: $\rightarrow_2$ IS NOETHERIAN.

THE PROOF IS IN TWO PARTS. WE FIRST GIVE A FUNCTION $E: \text{TERMS} \rightarrow N\text{-TUPLES OF POSITIVE INTEGERS}$.

WE SHOW THAT $T \xrightarrow{*} T' \Rightarrow E(T) > E(T')$, WHERE $>$ IS LEXICOGRAPHIC ORDERING. FOR THE SECOND PART, WE SHOW THAT FOR EVERY T THERE IS AN i S.T.

$T \xrightarrow{*} T' \Rightarrow E(T)$ IS A TUPLE OF LENGTH i OR LESS.

THUS, FOR ANY T , ANY SEQUENCE $T \rightarrow T_1 \rightarrow T_2$ IS WELL-FOUNDED, AS $E(T) > E(T_1) > E(T_2)$ IS WELL-FOUNDED.

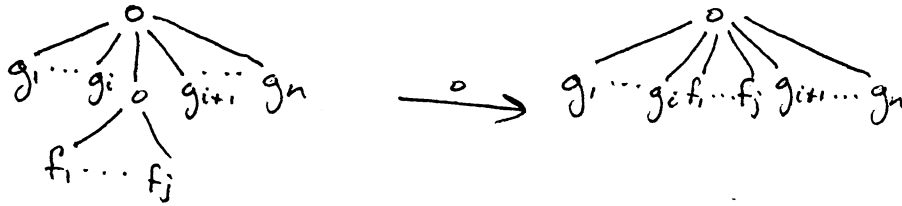
THEREFORE, $\rightarrow_2$ IS NOETHERIAN.

DEFINITIONS

- A TUPLE IS A SEQUENCE $\langle i_1, \dots, i_n \rangle$ WHERE $i_j \geq 0 \wedge i_n > 0$, i_j INTEGERS.
- IF $n \leq m$, THEN
$$\langle i_1, \dots, i_n \rangle + \langle j_1, \dots, j_m \rangle = \langle i_1 + j_1, \dots, i_n + j_n, j_{n+1}, \dots, j_m \rangle$$
- $\langle i_1, \dots, i_n \rangle > \langle j_1, \dots, j_m \rangle \iff \exists k$ s.t. $\forall p < k$ $(i_p = j_p \wedge i_k > j_k) \vee (i_p = j_p \wedge n > m = k)$
- THE DEPTH OF T/U IS $|U|$, THE LENGTH OF U .

WE WANT E TO BE INSENSITIVE TO ASSOCIATIVE TRANSFORMATIONS AS $\rightarrow_\alpha$ IS. THEREFORE, WE FIRST "MASK" THESE TRANSFORMATIONS:

DEFINE A RELATION $\xrightarrow{o}$ WITH RULE

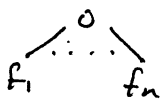


LEMMA: $\xrightarrow{o}$ IS CR AND NOETHERIAN.

LET $\hat{T}$ BE THE $\xrightarrow{o}$ -NORMAL FORM OF T .

$E(T) = E'(\hat{T})$ WHERE

$E'(\hat{T}) = \sum_{u \in N_T^*} e(T/u)$ WHERE

if T/u is  then $e(T/u) = \overset{|u|}{\langle 0, \dots, 0, n \rangle}$

if T/u is INTSTO then $e(T/u) = \langle 1 \rangle + E(\text{for: } \langle \text{IND}, \text{id} \rangle)$

if T/u is distr then $e(T/u) = \overset{|u|}{\langle 0, \dots, 0, 1 + E(\text{for: } \langle \text{---rhs of \#12---} \rangle)}$

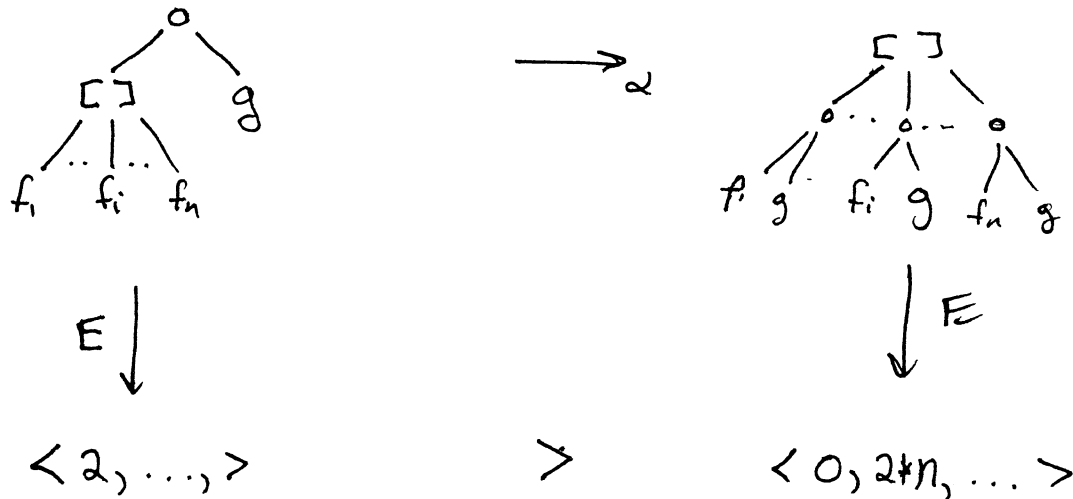
if T/u is distl then $e(T/u) = \overset{|u|}{\langle 0, \dots, 0, 1 + E(\text{rhs of \#13}) \rangle}$

if T/u is app then $e(T/u) = 0$ if parent is not con else $e(T/u) = \overset{|u|-1}{\langle 0, \dots, 4 \rangle}$

$e(T)$ IS 0 IF ONE OF THESE CASES DOES NOT APPLY.

LEMMA $T \xrightarrow{2} T' \Rightarrow E(T) > E(T')$

WE OMIT THE PROOF, AND GIVE ONLY AN EXAMPLE:



FOR THE SECOND PART OF THE LEMMA, NAMELY THAT TUPLES $E(T)$ ARE BOUNDED IN LENGTH FOR ANY SEQUENCE $E(T), E(T_2), \dots$ WHERE $T \rightarrow T_2 \rightarrow \dots$, NOTE THAT THE LENGTH OF THE TUPLE IS PROPORTIONAL TO THE DEPTH OF THE TERM T_i .

LEMMA FOR ANY T THERE IS A NUMBER n S.T.

$T \xrightarrow{*} T_i \Rightarrow \text{depth}(T_i) \leq n.$

PROOF: BY INDUCTION ON THE DEPTH OF T .

